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A COUNTERPART OF FAXÉN'S FORMULA IN POTENTIAL FLOW

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Abstract—An expression is derived for the acceleration of a sphere in a potential fluid in terms of the local incident flowfield, without posing any restriction upon the latter. The expression is exact, i.e. the effect of all spatial derivatives of the incident velocity field is taken into account.

1. INTRODUCTION

It is well known that the velocity V of a body moving in an otherwise undisturbed potential fluid of infinite extent is constant in absence of external forces (d'Alemberts paradox). It is furthermore well known that a massless sphere reacts to a uniform acceleration $\mathrm{d}U/\mathrm{d}t$ of the fluid according to

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 3\left(1 - 2\frac{\rho_b}{\rho}\right)^{-1}\frac{\mathrm{d}U}{\mathrm{d}t}, \qquad [1.1]$$

ρ_b and ρ being the mass densities of the body and the fluid, respectively. In [1.1] differentation of the uniform velocity U is unambiguously defined since U is a function of time only. When obstacles, other than the sphere itself, occur in the fluid the incident flowfield is not uniform anymore and the effect of the spatial derivatives of the incident velocity has to be considered. The pertaining generalization of [1.1] is far from obvious. There is an approximate result by Voinov (Voinov 1973) for a massless sphere which, written for a nonzero body mass, reads as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 3\left(1 - 2\frac{\rho_b}{\rho}\right)^{-1}\left(\frac{\partial u_0}{\partial t} + u \cdot \nabla u_0\right). \qquad [1.2]$$

Voinov's result would imply that the simple time derivative in [1.1] has to be replaced by the material derivative of the local incident fluid velocity $u_0 = \nabla\phi_0$. The fact that Voinov's result is not generally accepted and is moreover an approximation were reasons for us to reconsider the problem. In Sec. 2 we will derive the following extension of [1.2]:

$$\left(1 - 2\frac{\rho_b}{\rho}\right)\frac{\mathrm{d}V}{\mathrm{d}t} = 3\left(\frac{\partial u_0}{\partial t} + u_0 \cdot \nabla u_0\right) + 6\sum_{n=2}^{\infty}\frac{n!a^{2n-2}}{(n+1)!(2n-1)(2n-3)\cdots 3.1}\frac{\partial^n\phi_0}{\partial x^{\alpha_1}\cdots\partial x^{\alpha_n}}\frac{\partial^n u_0}{\partial x^{\alpha_1}\cdots\partial x^{\alpha_n}}. \qquad [1.3]$$

Voinov's approximation [1.2] represents the effect of the linear approximation to the incident velocity field and the correction, given by [1.3], is connected with the higher order nonuniformities. It is easy to see that the largest of the terms of the series of [1.3] is $\mathrm{O}(a^2/L^2)$ smaller than the first two "convective" terms, a being the radius of the sphere and L the length scale for variations in the incident velocity. As the latter is caused by the presence of other objects in the fluid its length scale is of the same order of magnitude as the distance from the sphere to these objects. Apparently Voinov's approximation is good as long as this distance is large compared to the sphere radius.

To apply [1.3] unambiguously to a given situation the incident flowfield must be defined rigorously. The definition presents itself if we imagine all boundaries in the fluid to be replaced by systems of singularities, a common abstraction in potential as well as Stokes flow. The flow incident at the sphere is precisely the difference between the total flow and the flow due to the singularities within the sphere, in other words, the part of the total flowfield which is regular at the location of the sphere. The concept of an incident flowfield enables us to formulate certain characteristic of the body, such as the acceleration, in terms of a local variable which represents the effect of other boundaries in the fluid as a whole. For an actual situation, i.e. for given positions and velocities of the boundaries, all singularities and thereby the incident flowfield can be determined by the method of reflections (Milne–Thompson 1938) although the calculations are feasible only for simple geometries.

It is interesting to compare [1.3] to an expression for the velocity of a sphere in Stokes flow obtained by Faxén many years ago (Oseen 1927):

$$V = u_0 + \frac{1}{6}a^2\nabla^2 u_0. \qquad [1.4]$$

[1.3] and [1.4] can be regarded as counterparts: both express the relevant dynamical variable in terms of the local incident flowfield. That the acceleration is the relevant variable for potential flow is not surprising. Indeed, in an undisturbed fluid the velocity of the sphere would remain constant and could be attributed an arbitrary value. Nonuniformities of the incident flow alter this arbitrary velocity and therefore determine its rate of change. In Stokes flow on the other hand the velocity of a moving body is not arbitrary but determined by the flow conditions. This is connected with the dissipative character of Stokes flow.

Higher order derivatives than the second do not appear in [1.4]: the derivatives of order four, six, etc. vanish as follows directly from the Stokes equations (odd order derivatives do not occur anyway). It is furthermore noteworthy that [1.3], as opposed to [1.4], is nonlinear. This is a consequence of the fact that in potential flow the pressure depends nonlinearly on the velocity while Stokes flow is an entirely linear phenomenon.

2. THE ACCELERATION AS A FUNCTION OF THE INCIDENT FLOWFIELD

The acceleration of a body immersed in a fluid is found by dividing the force $-\int p\,\mathbf{n}\,dS$ ($\mathbf{n}$ is the outward pointing unit vector normal to the body surface), exerted by the fluid on the body, by the mass of the body. Point of departure is therefore Bernoulli's equation

$$-\rho^{-1}p = \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2.$$

To express p in terms of the incident flowfield the total velocity potential ϕ has to be written in terms of the incident velocity potential ϕ_0. Since by definition ϕ_0 is regular at the location of the sphere it can be expanded in a Taylor series at the center $\mathbf{q}$ of the sphere:

$$\phi_0(\mathbf{r}, t) = \sum_{n=0}^{\infty}\frac{1}{n!}(x-q)^{\alpha_1}\cdots(x-q)^{\alpha_n}\left.\frac{\partial^n\phi_0}{\partial x^{\alpha_1}\cdots\partial x^{\alpha_n}}\right|_{\mathbf{r}=\mathbf{q}}, \qquad [2.1]$$

where $\mathbf{r} = (x^1, x^2, x^3)$ and $\mathbf{q} = (q^1, q^2, q^3)$.

It turns out that the subsequent calculations pass off most efficiently in a frame of reference with its origin fixed to the center of the sphere. Therefore we start from an incident velocity potential given in the moving (noninertial) frame and derive an expression for the force experienced by the sphere in this frame. After that we will transfer that expression to the inertial frame. The Taylor expansion of ϕ_0 in the moving frame remains essentially given by [2.1] but it is now centered at the origin of the frame. Still denoting the spatial coordinates in the moving frame by x^1, x^2, and x^3 we write instead of [2.1]:

$$\phi_0(\mathbf{r}, t) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{\alpha_1} \cdots x^{\alpha_n} \frac{\partial^n \phi_0}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_n}} \bigg|_{\mathbf{r}=0}. \qquad [2.2]$$

In what follows we will write $D^{\alpha_1 \cdots \alpha_n}\phi_0$ for $\partial^n \phi_0 / \partial x^{\alpha_1} \cdots \partial x^{\alpha_n} |_{\mathbf{r}=0}$, while $x^{\alpha_1} \cdots x^{\alpha_n} D^{\alpha_1 \cdots \alpha_n}\phi_0$, the nth spherical harmonic belonging to ϕ_0, is denoted by Q_n or, if necessary, by $Q_n(\phi_0)$.

In the frame fixed to the sphere the total velocity potential obviously has to satisfy the condition $\partial\phi/\partial r = 0$ at $r = a$. It is easy to verify that ϕ, given by

$$\phi(\mathbf{r}, t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(1 + \frac{n}{n+1}\frac{a^{2n+1}}{r^{2n+1}}\right) Q_n, \qquad [2.3]$$

is harmonic and satisfies the boundary condition at $r = a$. According to [2.3] $\phi(\mathbf{r}, t)$ is the sum of the incident potential $\phi_0(\mathbf{r}, t)$ and the potential due to a set of singularities located at the origin. These singularities may be identified as multipoles of increasing order, starting with a dipole for $n = 2$. Note however that the relation between the strength of these multipoles and the derivatives of the incident potential is far from simple (the velocity potential of a multipole of order n and strength P_n is given by $(-1)^n P_n D^{\alpha_1 \cdots \alpha_n}(r^{-1})$ (Hobson 1955).

We now proceed with the integration of $\frac{1}{2} | \nabla\phi |^2$ on the surface of the sphere. Evaluate the derivative of [2.3] at $r = a$,

$$\frac{\partial\phi}{\partial x^\theta} = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)!}\left(\frac{\partial Q_n}{\partial x^\theta} - na^{-2}Q_n x^\theta\right)$$

to express $\frac{1}{2} | \nabla\phi |^2$ in terms of the spherical harmonics of ϕ_0:

$$\frac{1}{2}\int | \nabla\phi |^2 n^\alpha \mathrm{d}S = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} \frac{(2n+1)(2k+1)}{(n+1)!(k+1)!} \cdot \int\left(\frac{\partial Q_n}{\partial x^\theta} - na^{-2}Q_n x^\theta\right)\left(\frac{\partial Q_k}{\partial x^\theta} - ka^{-2}Q_k x^\theta\right) n^\alpha \,\mathrm{d}S.$$

Using the identity $x^\theta(\partial Q_n/\partial x^\theta) = nQ_n$, which holds for homogeneous polynomials of degree n, we can write

$$\frac{1}{2}\int | \nabla\phi |^2 n^\alpha \mathrm{d}S = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(2n+1)(2k+1)}{(n+1)!(k+1)!} \cdot \int(\nabla Q_n \cdot \nabla Q_k - nka^{-2}Q_n Q_k) n^\alpha \mathrm{d}S. \qquad [2.4]$$

Since

$$\frac{\partial Q_n}{\partial x^\theta} = \frac{\partial(x^{\alpha_1} \cdots x^{\alpha_n})}{\partial x^\theta} D^{\alpha_1 \cdots \alpha_n}\phi_0 = nx^{\alpha_1} \cdots x^{\alpha_{n-1}} D^{\alpha_1 \cdots \alpha_{n-1}}(D^\theta\phi_0),$$

$\partial Q_n / \partial x^\theta$ is apparently n times the $(n-1)$th spherical harmonic of $D^\theta\phi_0$ and we write [2.4] as

$$\frac{1}{2}\int|\nabla\phi|^2 n^\alpha \mathrm{d}S = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(2n+1)(2k+1)}{(n+1)!(k+1)!}nk \cdot \int[Q_{n-1}(D^\theta\phi_0)Q_{k-1}(D^\theta\phi_0) - a^{-2}Q_n(\phi_0)Q_k(\phi_0)]n^\alpha \mathrm{d}S$$

in an obvious notation.

This expression is symmetric in n and k and so

$$\begin{aligned}\frac{1}{2}\int|\nabla\phi|^2 n^\alpha \mathrm{d}S &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n+1)^2 n^2}{((n+1)!)^2}\int[Q^2_{n-1}(D^\theta\phi_0) - a^{-2}Q^2_n(\phi_0)]n^\alpha \mathrm{d}S \\ &+ \sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty}\frac{(2n+1)(2k+1)}{(n+1)!(k+1)!}nk \\ &\cdot \int[Q_{n-1}(D^\theta\phi_0)Q_{k-1}(D^\theta\phi_0) - a^{-2}Q_n(\phi_0)Q_k(\phi_0)]n^\alpha \mathrm{d}S.\end{aligned} \qquad [2.5]$$

Let us first evaluate

$$\int Q_n(\phi_0)Q_k(\phi_0)n^\alpha \mathrm{d}S = D^{\alpha_1\cdots\alpha_n}\phi_0 \cdot D^{\beta_1\cdots\beta_k}\phi_0\int x^{\alpha_1}\cdots x^{\alpha_n}x^{\beta_1}\cdots x^{\beta_k}n^\alpha \mathrm{d}S.$$

Apply Gauss' theorem to the surface integral:

$$\begin{aligned}&\int Q_n(\phi_0)Q_k(\phi_0)\,n^\alpha \mathrm{d}S \\ &= D^{\alpha_1\cdots\alpha_n}\phi_0 \cdot D^{\beta_1\cdots\beta_k}\phi_0\int\frac{\partial(x^{\alpha_1}\cdots x^{\alpha_n}x^{\beta_1}\cdots x^{\beta_k})}{\partial x^\alpha}\mathrm{d}V \\ &= nD^{\alpha_1\cdots\alpha_{n-1}\alpha}\phi_0 \cdot D^{\beta_1\cdots\beta_k}\phi_0\int x^{\alpha_1}\cdots x^{\alpha_{n-1}}x^{\beta_1}\cdots x^{\beta_k}\mathrm{d}V \\ &\quad + kD^{\alpha_1\cdots\alpha_n}\phi_0 \cdot D^{\beta_1\cdots\beta_{k-1}\alpha}\phi_0\int x^{\alpha_1}\cdots x^{\alpha_n}x^{\beta_1}\cdots x^{\beta_{k-1}}\mathrm{d}V.\end{aligned} \qquad [2.6]$$

Both integrands in [2.6] are the product of $k + n - 1$ components of the coordinate vector $\mathbf{r}$. In spherical coordinates r, θ, and ω the integrands could be split up in a factor r^{k+n-1} and a factor depending only on the angles θ and ω. As the boundary of the integration region is spherical the integration of both factors can be carried out independently. It follows from this observation that, since

$$\int_0^a r^{k+n-1}r^2\mathrm{d}r = \frac{a^{k+n+2}}{k+n+2}.$$

the following two identities hold:

$$\int x^{\alpha_1}\cdots x^{\alpha_{n-1}}x^{\beta_1}\cdots x^{\beta_k}\mathrm{d}V = \frac{a}{k+n+2}\int x^{\alpha_1}\cdots x^{\alpha_{n-1}}x^{\beta_1}\cdots x^{\beta_k}\mathrm{d}S,$$

$$\int x^{\alpha_1}\cdots x^{\alpha_n}x^{\beta_1}\cdots x^{\beta_{k-1}}\mathrm{d}V = \frac{a}{k+n+2}\int x^{\alpha_1}\cdots x^{\alpha_n}x^{\beta_1}\cdots x^{\beta_{k-1}}\mathrm{d}S.$$

Inserting these identities into [2.6], one finds

$$\int Q_n(\phi_0)Q_k(\phi_0)n^\alpha \mathrm{d}S$$

$$= \frac{a}{k+n+2}\int [nQ_{n-1}(D^\alpha\phi_0)Q_k(\phi_0) + kQ_n(\phi_0)Q_{k-1}(D^\alpha\phi_0)]\mathrm{d}S. \quad [2.7]$$

Next we apply basically the same reduction technique to $\int Q_{n-1}(D^\alpha\phi_0)Q_k(\phi_0)\mathrm{d}S$:

$$\int Q_{n-1}(D^\alpha\phi_0)Q_k(\phi_0)\mathrm{d}S$$

$$= D^{\alpha_1\cdots\alpha_{n-1}\alpha}\phi_0 \cdot D^{\beta_1\cdots\beta_k}\phi_0 \int x^{\alpha_1}\cdots x^{\alpha_{n-1}}x^{\beta_1}\cdots x^{\beta_k}\mathrm{d}S$$

$$= aD^{\alpha_1\cdots\alpha_{n-1}\alpha}\phi_0 \cdot D^{\beta_1\cdots\beta_k}\phi_0 \int \frac{\partial(x^{\alpha_1}\cdots x^{\alpha_{n-1}}x^{\beta_1}\cdots x^{\beta_{k-1}})}{\partial x^{\beta_k}}\mathrm{d}V$$

$$= (n-1)a\,D^{\alpha_1\cdots\alpha_{n-1}\alpha}\phi_0 \cdot D^{\beta_1\cdots\beta_{k-1}\alpha_{n-1}}\phi_0 \int x^{\alpha_1}\cdots x^{\alpha_{n-2}}x^{\beta_1}\cdots x^{\beta_{k-1}}\mathrm{d}V$$

$$+ (k-1)a\,D^{\alpha_1\cdots\alpha_{n-1}\alpha}\phi_0 \cdot D^{\beta_1\cdots\beta_{k-2}\beta_k\beta_k}\phi_0 \int x^{\alpha_1}\cdots x^{\alpha_{n-1}}x^{\beta_1}\cdots x^{\beta_{k-2}}\mathrm{d}V.$$

The last term vanishes since ϕ_0 is harmonic. The volume integral in the last term but one can be converted to a surface integral in the same way as before yielding

$$\int Q_{n-1}(D^\alpha\phi_0)Q_k(\phi_0)\,\mathrm{d}S = \frac{a^2(n-1)}{n+k}D^{\alpha_1\cdots\alpha_{n-2}}D^{\alpha_{n-1}\alpha}\phi_0$$

$$\cdot D^{\beta_1\cdots\beta_{k-1}}D^{\alpha_{n-1}}\phi_0 \int x^{\alpha_1}\cdots x^{\alpha_{n-2}}x^{\beta_1\cdots\beta_{k-1}}\mathrm{d}S \quad [2.8]$$

$$= \frac{(n-1)a^2}{n+k}\int Q_{n-2}(D^{\alpha_{n-1}\alpha}\phi_0)\,Q_{k-1}(D^{\alpha_{n-1}}\phi_0)\,\mathrm{d}S.$$

[2.8] Represents the first step of a systematic reduction process for $\int Q_{n-1}(D^\alpha\phi_0)Q_k(\phi_0)\,\mathrm{d}S$. After i reduction steps one obtains

$$\int Q_{n-1}(D^\alpha\phi_0)Q_k(\phi_0)\,\mathrm{d}S = \frac{(n-1)(n-2)\cdots(n-i)a^{2i}}{(n+k)(n+k-2)\cdots(n+k-2i)}$$

$$\cdot\int Q_{n-i-1}(D^{\alpha_{n-i}\cdots\alpha_{n-1}\alpha}\phi_0)\,Q_{k-i}(D^{\alpha_{n-i}\cdots\alpha_{n-1}}\phi_0)\,\mathrm{d}S. \quad [2.9]$$

The reduction proces is continued until $n-i-1$ or $k-i$ is zero. We are then left with an integrand $x^{\alpha_1}\cdots x^{\alpha_{n-k-1}}$ (if $n-1>k$), $x^{\beta_1}\cdots x^{\beta_{k-n+1}}$ (if $n-1<k$) or 1 (if $n-1=k$). In the first two cases further reduction leads to the result $\int Q_{n-1}(D^\alpha\phi_0)Q_k(\phi_0)\,\mathrm{d}S = 0$ for a harmonic function ϕ_0. This corresponds to the well known fact that the integral on a sphere of two spherical harmonics of different degree is zero (Hobson 1955). For $i=k=n-1$ [2.9] becomes

$$\int Q_{n-1}(D^\alpha\phi_0)Q_{n-1}(\phi_0)\,\mathrm{d}S = 4\pi\frac{(n-1)!a^{2n}}{(2n-1)(2n-3)\cdots 3.1} \quad [2.10]$$

$$D^{\alpha_1\cdots\alpha_{n-1}\alpha}\phi_0 \cdot D^{\alpha_1\cdots\alpha_{n-1}}\phi_0.$$

The second term of [2.7] can be evaluated in the same spirit:

$\int Q_n(\phi_0) Q_{k-1}(D^\alpha \phi_0)\, dS$ is nonzero only when $k = n + 1$ and

$$\int Q_n(\phi_0) Q_n(D^\alpha \phi_0) = 4\pi \frac{n!\, a^{2n+2}}{(2n+1)(2n-1)\cdots 3.1} D^{\alpha_1 \cdots \alpha_n \alpha} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_n} \phi_0,$$

which follows from replacing $n - 1$ by n in [2.10].

Using this result and [2.10] in [2.7] we find

$$\int Q_n(\phi_0) Q_k(\phi_0)\, n^\alpha\, dS = 4\pi \frac{n!\, a^{2n+1}}{(2n+1)(2n-1)\cdots 3.1} \cdot \left[\delta_{k,n-1} D^{\alpha_1 \cdots \alpha_{n-1} \alpha} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_{n-1}} \phi_0 + \delta_{k,n+1} \frac{n+1}{2n+3} a^2 D^{\alpha_1 \cdots \alpha_n \alpha} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_n} \phi_0 \right]. \tag{2.11}$$

Replacing n by $n - 1$, k by $k - 1$, and ϕ_0 by $D^\theta \phi_0$ in [2.11] one finds

$$\int Q_{n-1}(D^\theta \phi_0) Q_{k-1}(D^\theta \phi_0)\, n^\alpha\, dS = 4\pi \frac{(n-1)!\, a^{2n-1}}{(2n-1)(2n-3)\cdots 3.1} \cdot \left[\delta_{k,n-1} D^{\alpha_1 \cdots \alpha_{n-2} \alpha \theta} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_{n-2} \theta} \phi_0 + \delta_{k,n+1} \frac{n a^2}{2n+1} D^{\alpha_1 \cdots \alpha_{n-1} \alpha \theta} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_{n-1} \theta} \phi_0 \right].$$

We are now in the position to evaluate $\frac{1}{2} \int |\nabla \phi|^2 n^\alpha dS$. The single series in [2.5] vanishes because of the orthogonality property of the spherical harmonics. Taking furthermore into account that only the values $k = n + 1$ contribute to [2.5], one finds after some elementary algebra that

$$\frac{1}{2} \int |\nabla \phi|^2 n^\alpha dS = 4\pi \sum_{n=1}^{\infty} \cdot \frac{n a^{2n+1}}{(n+1)!(2n-1)(2n-3)\cdots 3.1} D^{\alpha_1 \cdots \alpha_n} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_n \alpha} \phi_0. \tag{2.12}$$

Remains to integrate $\partial \phi / \partial t = D_t \phi$. Differentiate [2.2] with respect to time:

$$\frac{\partial \phi}{\partial t} = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)!} x^{\alpha_1} \cdots x^{\alpha_n} D_t D^{\alpha_1 \cdots \alpha_n} \phi_0.$$

It is easy to see that only the first term of the series survives integration:

$$\int \frac{\partial \phi}{\partial t} n^\alpha dS = \frac{3}{2} \tau D_t D^\alpha \phi_0,$$

with $\tau = (4\pi/3)\, a^3$. Adding this expression to [2.12] we have the force on the sphere in the reference frame fixed to the sphere:

$$- \int p n^\alpha dS = \frac{3}{2} \rho \tau\, D_t D^\alpha \phi_0 + 4\pi\rho \sum_{n=1}^{\infty} \frac{n a^{2n+1}}{(n+1)!(2n-1)\cdots} D^{\alpha_1 \cdots \alpha_n} \phi_0 \cdot D^{\alpha_1 \cdots \alpha_n \alpha} \phi_0.$$

To transform this expression to an inertial frame, with respect to which the velocity V of the sphere is measured, the pressure gradient and the velocity potential have to be modified by an amount of $\rho(dV/dt)$ and $\mathbf{V} \cdot \mathbf{r}$, respectively. We obtain after the pertaining transformation of the time derivative of ϕ_0:

$$-\int pn^\alpha \mathrm{d}S = -\frac{1}{2}\rho\tau\frac{\mathrm{d}V^\alpha}{\mathrm{d}t} + \frac{3}{2}\rho\tau\, D_t D^\alpha\phi_0$$
$$+ 4\pi\rho\sum_{n=1}^{\infty}\frac{na^{2n+1}}{(n+1)!(2n-1)(2n-3)\cdots 3.1} D^{\alpha_1\cdots\alpha_n}\phi_0 \cdot D^{\alpha_1\cdots\alpha_n\alpha}\phi_0,$$

where all variables are now defined in the inertial frame. Putting the body mass equal to $\rho_b\tau$ we find from the condition $-\int pn^\alpha \mathrm{d}S = \rho_b\tau\dot{V}^\alpha$ that

$$\frac{\mathrm{d}V^\alpha}{\mathrm{d}t} = \left(1 - 2\frac{\rho_b}{\rho}\right)^{-1}\left[3\left(\frac{\partial u_0^\alpha}{\partial t} + u_0^{\alpha_1}D^{\alpha_1}u_0^\alpha\right)\right.$$
$$\left. + 6\sum_{n=2}^{\infty}\frac{na^{2n-2}}{(n+1)!(2n-1)\cdots 3.1} D^{\alpha_1\cdots\alpha_n}\phi_0 \cdot D^{\alpha_1\cdots\alpha_n\alpha}\phi_0\right],$$

which is [1.3] in a slightly different notation.

The material derivative $\partial u_0/\partial t + u_0^{\alpha_1}D^{\alpha_1}u_0$ can be written eventually as ∇p_0, the gradient of the incident pressure. Body forces can be incorporated by discounting them in the incident pressure.

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